Indian J. Phys. 43, 519—527 (1969)

Propagation of shock waves in earth's atmosphere

By V. P. Singh

Defence Science Laboratory, Delhi†

(*Received* 28 *July* 1969)

A plane shock wave as it propagates vertically upward in the static atmosphere of earth is studied. Variation of Mach number and velocity of the shock is obtained analytically by using Whitham's Rule. It is found that shock velocity increases as the shock propagates in an atmosphere with decreasing density normal to the plane of the shock front.

Introduction

Problem of propagation of shock waves in non-uniform medium with various density distributions has been discussed by different authors, such as, Kopal (1956), Carrus *et al* (1951), Grover & Hardy (1966). These authors have used the technique of similarity solutions and have found the behaviour of the fluid flow in the presence of shock waves.

Applying Whitham's method (1958) to the propagation of shock waves in a non-homogeneous medium, Bhatnagar & Sachdev (1966) have obtained the differential relation between the Mach number, pressure and density. This differential equation was integrated numerically for different stellar models.

In the present paper we have also used the Whitham's rule and discussed the problem of propagation of shock waves in the earth's atmosphere. With minor changes the experimental data for temperature distribution given by Mitra (1952) has been used. It is assumed that the gravity is varying as

$$g_0 = g_s\left(\frac{R_0}{R_0+x_0}\right)^2 \qquad ...(1)$$

where g_s is the value of g_0 at the surface of the earth, R_0 is the radius of the earth and x_0 is the vertical distance from the surface of the earth. Using different temperature distributions of the earth's atmosphere, the variation of the Mach number of the shock is obtained. In the latter part, the results are combined to find the variation of shock velocity as the shock propagates in the atmosphere. Variation of shock velocity is shown in figure 1. It is found that the shock velocity increases as the shock propagates into the medium which becomes rarer and rarer with the distance measured from the surface of the earth.

As the shock velocity increases with the distance, high velocity of the shock causes the fluid velocity to be greater than the escape velocity in

† Present address : Terminal Ballistic Research Laboratory, Chandigarh.

the upper atmosphere. But at such a height, the density of the fluid is so small that very small quantity of the fluid escapes from the earth's gravity.

This study of the atmosphere is somewhat approximate, partly because the temperature variations in the atmosphere are not exactly what we have considered, partly because the method used is appoximate. Moreover the effects of magnetic field, turbulent motion of gases in the atmosphere and solar radiations, which have not been taken into acount, are disturbing the atmosphere. But in all, this study leads us to a physical picture of the shock wave propagation in the atmosphere of the earth.

Formulation of the problem

We assume that there is an intense explosion at the surface of the earth, due to which a strong shock wave propagates in the atmosphere. As the radius of the earth is very large in comparision to the height of the atmosphere, therefore it can be assumed that the layers of the atmosphere are planer and the path of shock wave is one dimensional. Taking x-axis to be normal to the surface of the earth, let p_s, ρ_s, T_s, g_s and p_0, ρ_0, T_0, g_0 be pressure, density, absolute temperature and acceleration due to gravity at the surface of the earth and at a distance x_0 from the surface. The gravity g_0 at a distance x_0 is assumed (Mitra 1952) to be as given in equation. The pressure variations in equilibrium conditions is given by,

$$\frac{dp_0}{p_0} = -\frac{g_0}{RT_0}\, dx_0 \qquad \text{... (2)}$$

where R is the gas constant. We take α a standard constant, having dimensions as inverse of a distance, as

$$\alpha = \frac{g_s}{RT_s} \qquad \text{...(3)}$$

and u, p, ρ, x, t, and g as the dimensionless fluid velocity, pressure, density, distance, time and acceleration due to gravity, as

$$u = \sqrt{\gamma}\, u_0/c_s\,,\; p = p_0/p_s\,,\; \rho = \rho_0/\rho_s\,,\; x = \alpha x_0\,,$$

$$\bar{R} = \alpha R_0,\; t = \frac{\alpha c_s t_0}{\sqrt{\gamma}} \text{ and } g = g_0/g_s \qquad \text{...(4)}$$

where $\quad c_s^2 = \gamma p_s/\rho_s$

The equations of motion in dimensionless parameters can be written as

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \left(\frac{\bar{R}}{\bar{R}+x}\right)^2 = 0 \qquad \ldots(5)$$

$$\frac{\partial}{\partial t}(p\rho^{-\gamma}) + u\frac{\partial}{\partial x}(p\rho^{-\gamma}) = 0.$$

When the shock produced on the earth reaches a distance $x = r$ where u, p, ρ, are the fluid velocity, presure and density and if u_2, p_2, ρ_2, are the values of u, p, ρ, behind the shock front, Rankine-Hugoniot relations are given by,

$$u_2(r, t) = \frac{2c}{(\gamma+1)}\{M - M^{-1}\}$$

$$p_2(r, t) = \frac{\gamma p}{(\gamma+1)} f(M) \qquad \ldots(6)$$

$$\rho_2(r, t) = \frac{(\gamma+1)\rho M^2}{g(M)}$$

where

$$f(M) = \left\{2M^2 - \frac{\gamma-1}{\gamma}\right\}$$

$$g(M) = \{2 + (\gamma - 1)M^2\} \qquad \ldots(7)$$

and

$$c^2 = \gamma p/\rho.$$

We apply Whitham's Rule to the flow paramaters behind the shock front. Equation of motion along the positive characteristic axis, just behind the shock front is,

$$dp_2 + \rho_2 c_2 du_2 + \frac{\rho_2 c_2}{u^2 + c^2}\left(\frac{\bar{R}}{\bar{R}+r}\right)^2 dr = 0 \qquad \ldots(8)$$

Substituting values of parameters u_2, ρ_2 and c_2 from relations (6) and (7) in relation (8), we get after some simplifications,

$$2\{(M^2+1)\,h(M) + 2M^2\}\frac{dM}{M} + f(M)\frac{dp}{p} + 2(M^2-1)h(M)\frac{dc}{c}$$

$$+\left(\frac{\bar{R}}{\bar{R}+r}\right)^2 \frac{(\gamma+1)^2 M^2 h(M)dr}{c^2\{2(M^2-1)+\sqrt{r+g}\}} = 0 \qquad \ldots(9)$$

where c is the dimensionless sound velocity and r is the distance of the shock front from the surface of the earth. The equation (9) gives the relation between M, p and c. If the absolute temperature is given, one can find pressure from equation (2) and M can be calculated from the relation (9).

Now the atmosphere can be adiabatic or isothermal or the temperature may be monotonically increasing or decreasing in it. Therefore, the relation (9) has been discussed for different cases.

Adiabatic Atmosphere

The atmosphere of the earth is extremely transparent to the radiations from the sun, and is hardly heated by it. Thus almost all the radiation from the sun falls on the surface of the earth. The radiation falling on the surface of the earth is reflected back to the atmosphere in the form of infrared rays. The infrared rays are absorbed by the carbon dioxide and water vapour in the lower part of the atmosphere. Thus near the surface of the earth, the temperature falls rapidly with the height. This rapid fall produces instability in the density of the atmosphere and thus there are strong air currents in this region. They stabilize the fluctuations in the temperature. As the rate of conduction of heat in the gases is very small, therefore an adiabatic equilibrium condition is set up. This state occurs in the troposphere. For adiabatic atmosphere the dimensionless pressure, density, temperature and sound velocity are given as,

$$
\begin{aligned}
p &= \left[1 - \frac{\gamma-1}{\gamma}\,\frac{\bar{R}\,x}{\bar{R}+x}\right]^{\frac{\gamma}{(\gamma-1)}} \\
\rho &= \left[1 - \frac{\gamma-1}{\gamma}\,\frac{\bar{R}\,x}{\bar{R}+x}\right]^{\frac{1}{(\gamma-1)}} \qquad \ldots(10) \\
T &= \left[1 - \frac{\gamma-1}{\gamma}\,\frac{\bar{R}x}{\bar{R}+x}\right] \\
c^2 &= \gamma p/\rho,
\end{aligned}
$$

Substituting the value of p and c from (10) into (9), one gets after some simplifications at $x = r$,

$$
\frac{2}{M}\frac{dM}{dr} = K_3(M)\,\frac{(\bar{R}/(\bar{R}+r))^2}{\left[1 - \frac{\gamma-1}{\gamma}\,\frac{\bar{R}r}{\bar{R}+r}\right]} \qquad \ldots(11)
$$

where,

$$
K_3(M) = K_1(M) + \frac{\gamma-1}{\gamma}\,K_2(M) \qquad \ldots(12)
$$

$$K_1(M) = \frac{\left\{ f(M) - \dfrac{(\gamma+1)^2/\gamma M^2 h(M)}{\{2(M^2-1)+\sqrt{\gamma+g}\}} \right\}}{\{(M^2+1)h(M)+2M^2\}}$$

$$K_2(M) = \frac{2(M^2-1)h(M)}{\{(M^2+1)h(M)+2M^2\}} \qquad \text{...(13)}$$

From the numerical computations it is found that variation in $K_1(M)$ and $K_2(M)$ and therefore in $K_3(M)$ is small for values of M greater than 3. Therefore for the purpose of integration, we can take $K_3(M)$ outside the integral sign while integrating equation (11).

Integrating relation (11) between $r = 0$ to $r = r$ one gets,

$$M = M_s \left[1 - \frac{\gamma-1}{\gamma} \, \frac{\bar{R}r}{\bar{R}+r} \right]^{-\frac{\gamma K_3(M)}{2(\gamma-1)}} \qquad \text{...(14)}$$

and hence the shock velocity U can be obtained as,

$$U = \sqrt{\gamma} M_s \left[1 - \frac{\gamma-1}{\gamma} \, \frac{\bar{R}r}{\bar{R}+r} \right]^{-\left(\frac{\gamma K_3}{\gamma-1} - 1\right)/2} \qquad \text{...(15)}$$

The relations (14) and (15) give variation of Mach number and shock velocity in an adiabatic atmosphere. It is observed from these expressions that the adiabatic atmosphere is monotonically decreasing only for

$r < \dfrac{\gamma \bar{R}}{(\gamma-1)\bar{R}-\gamma}$ or $\lesssim \dfrac{\gamma}{\gamma-1} = 3.5$, for $\gamma = 1.4$. It is also observed that

Mach number and shock velocity increase as r increases from zero to 3.5. At $r = 3.5$, there is discontinuity, meaning that the atmosphere is unstable.

Isothermal Atmosphere

In case there are no external forces acting on the atmosphere, there will be no motion of the air. Since conduction of heat from one part of the atmosphere to the other is slow, in the absence of the external forces the atmosphere will attain uniform temperature after sufficient length of time. If T_1 be uniform dimensionless temperature, p_1 the pressure at $x = x_1$, the pressure at a distance x, $x \geqslant x_1$, is given by,

$$p = p_1 \exp\{-\bar{R}^2(x - x_1)/T_1(\bar{R}+x_1)(\bar{R}+x)\} \qquad \text{...(16)}$$

substituting the value of p in equation (9), one gets at $x = r$,

$$\frac{2}{M}\frac{dM}{dr} = \left(\frac{\bar{R}}{\bar{R}+r}\right)^2 \frac{K_1(M)}{T_1} \qquad \text{...(17)}$$

Integrating (17) taking $K_1(M)$ to be constant for the purpose of integrating, it is easy to get,

$$M = M_1 \exp\left\{ \frac{\bar{R}^2 K_1(M)(r - r_1)}{2T_1 \, (\bar{R} + r_1)(\bar{R} + r)} \right\} \qquad \ldots(18)$$

and the shock velocity as,

$$U = U_1 \exp\left\{ \frac{\bar{R}^2 K_1(M)\,(r - r_1)}{2T_1 \, (\bar{R} + r_1)(\bar{R} + r)} \right\} \qquad \ldots(19)$$

where M_1, U_1 are the Mach number and the shock velocity at $r = r_1$.

Medium with Monotonically Increasing Temperature

In the upper atmosphere, ultraviolet rays of the sun cause some of the air to ionise, which combines with oxygen and forms ozone. Ozone is found at the height of 10 km to 15 km from the surface of the earth. It absorbs the heat of the sun and thus the atmosphere is heated by it. The heat absorbed up to the height of 30 km is negligible, but beyond 30 km the temperature of the atmosphere starts increasing. Here gas is much rarefied. If β is the rate of increase of temperature per kilometer, T_2 the temperature at a distance x_2, the beginning of the hot layer, we have

$$T = T_2 + \frac{\beta}{\alpha T_s}(x - x_2) \qquad \ldots(20)$$

The dimensionless pressure at a distance x_2 from the surface of the earth is,

$$p = p_2 \left\{ \frac{T_2(\bar{R} + \bar{x})}{\bar{R}_1(T_2 + \delta\bar{x})} \right\}^{\frac{\bar{R}^2\delta}{(\bar{R}_1\delta - T_2)^2}} \exp\left\{ \frac{-\bar{x}\bar{R}_1}{(\bar{R}_1 + \bar{x})(T_2 - \bar{R}_1\delta)} \right\} \qquad (21)$$

where, $\delta = \dfrac{\beta}{\alpha T_s}$, $\bar{R}_1 = \bar{R} + x_2$, $\bar{x} = x - x_2$, –

Substituting $\dfrac{dp}{p}$ and $\dfrac{dc}{c}$ from (21) and (20) respectively, into (9

we get,

$$\frac{2}{M}\frac{dM}{dr} = \frac{\bar{R}^2 K_1(M)}{(T_2 + \delta\bar{r})(\bar{R}_1 + \bar{r})^2} + \frac{\delta K_2(M)}{2(T_2 + \delta\bar{r})} \qquad \ldots(22)$$

ntegrating (22) between $r = r_2$ to $r = r$, we get,

$$M = M_2 \left\{ \frac{\bar{R}_1(T_2 + \delta\bar{r})}{T_2(\bar{R}_1 + \bar{r})} \right\}^{\frac{\bar{R}^2 \delta K_1}{2(\bar{R}_1\delta - T_2)^2}} \left\{ \frac{T_2 + \delta\bar{r}}{T_2} \right\}^{-K_2/4}$$

$$\times \exp\left\{ \frac{\bar{R}^2 K_1 \bar{r}}{2\bar{R}_1(R_1 + \bar{r})\,(T_2 - \bar{R}_1\,\delta)} \right\} \qquad \ldots(23)$$

The relation (23) gives the variation of Mach number in hot layer. The shock velocity is obtained as,

$$U = \sqrt{\gamma} M (T_2 + \delta\bar{r})^{1/2} \qquad ..(24)$$

The hot layer extends from 30 to 50 km from the surface of the earth.

The General Problem

From the results so far discussed, one can find the variation of the Mach number and the shock velocity, when the shock propagates in the atmosphere of earth. We have considered a model in which the temperature decreases adiabatically from $x_0 = 0$ to $x_0 = 10$ km. Then it becomes constant until the height of 30 km is attained. Between 30 km to 50 km the temperature increases with gradient 5.5°K per km, K being the absolute temperature. From 50 to 55 km, temperature again remains constant and then upto 78 km it decreases with temperature gradient 9°K per km. Beyond this height it is assumed that the temperature increases continuously with temperature gradient 3.25°K per km. (Kolobkov 1960).

We assume that the shock wave propagates in the above mentioned atmosphere with initial Mach number 4 at the surface of the earth. The variation of shock velocity is computed from relation (15) for adiabatic atmosphere, from (19) for isothermal atmosphere and from (23) and (24) for atmosphere with increasing temperature and is shown in figure 1. It can be seen that the shock velocity increases as the shock propagates in the decreasing density medium. The variation of shock velocity is slower in isothermal medium, but is sharp in the rest of the medium. It is concluded that the variation of shock velocity mainly depends on the density of the medium and increases as the density decreases, and is not much effected by the variation of the temperature.

Escape of Gases from Atmosphere

When the shock moves in the gaseous medium, the fluid behind the shock is also set in motion. If the fluid velocity is greater than the escape

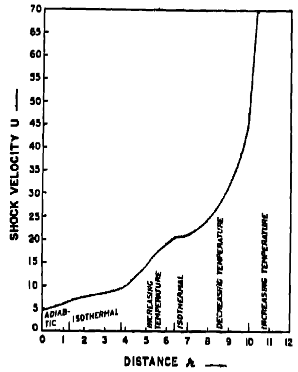

Figure 1. Variation of the shock velocity in the atmosphere.

velocity of the gas particles, the particles will escape the earth's gravitation and go into space. The escape velocity of the fluid particles is given by,

$$v^2_E = \frac{2g\bar{R}^2}{(\bar{R} + r)}$$

In figure 2 we have drawn the fluid velocity behind the shock and escape velocity versus distance from the surface of the earth. It is found that the fluid velocity is greater than the escape velocity at a distance $r = 10$. But at such a height the density of the air is very small and the amount

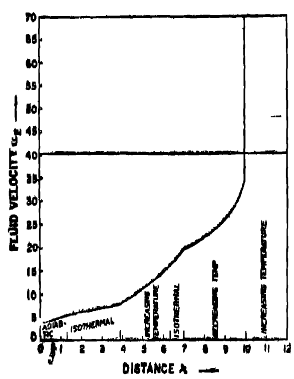

Figure 2. Variation of the fluid velocity behind the shock front in the atmosphere.

of the gas escaping in the space is negligible as compared to the huge mass of the atmosphere.

Conclusion

Although the model considered is idealized yet it leads us to some significant results. It is found that the strength of the shock goes on increasing as the atmosphere becomes rarefied. In isothermal part, the increase in shock velocity is small but is large in layers in which the temperature decreases. Above 78 km of height, we have assumed temperature to be increasing. In this region the shock becomes stronger and stronger as it propagates upwords. In ionosphere the gas is much rarefied and is ionized too. Is this region fluid velocity becomes greater than the escape velocity, so that some of the gas escapes from the earth's gravitation, though the amount of the gas escaped is very small as compared to the whole mass of the atmosphere.

Beauty of the method considered in this investigation is that it can be applied to the more complex problems of the earth's atmosphere. The method used is simple, although a bit approximate. By this method it is possible to get analytic relations for the Mach number and the shock velocity, which was not so easy by the earlier methods. It can be directly seen that Mach number and the shock velocity are variables, not constant as ahown in references. (Bhatnagar & Sachdev 1966 ; Carrus 1951 ; Kopal 1956).

The author is thankful to Dr. Prem Kumar, for his sincere guidance and to Dr. R. S. Srivastava and Dr. R. R. Agarwal, for their interest in the work. Thanks are also due to Dr. Kartar Singh, Director, D.S.L. Delhi, for his kind permission to publish the work.

References

Bhatnagar, P. L & Sachdev, P.L. 1966 *Nuovo Cimento*, **44**, 15.

Carrus, P.A., Fox, P. A, Kopal, Z & Haas, F 1951 *Ap. J.*, **113**,

Grover. R & Hardy, J.W. 1966 *Ap. J.*, **143** 193, 496, 48.

Kopal, Z. 1956 *Ap. J.*, **120**, 159.

Kolobkov, N., 1960 *Our Atmospheric Ocean*, Foreign Language Publishing House, Moscow, **41**.

Mitra, S K. 1952, *Upper Atmosphere*, Asiatic Society, Calcutta.

Whitham, G.B., 1958 *J. Fluid Mech.*, **4**, 337.